JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 16, No. 5, September–October 1993

State Equations for Maneuvering and Control of Flexible Bodies Using Quasimomenta

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This paper is concerned with the derivation of state equations suitable for the task of designing controls for the maneuvers of flexible bodies in space while controlling the elastic vibration. The equations are in terms of quasimomenta, defined as momenta corresponding to quasicoordinates, and are simpler in form than the equations in terms of quasicoordinates. A perturbation approach permits design of a dual-level control, a high-authority control for the rigid-body maneuvering and a low-authority control for the suppression of the elastic vibration.

I. Introduction

A PROBLEM of current interest is concerned with the maneuvering and control of flexible bodies. This problem arises in space applications and in robotics. The equations of motion for flexible bodies in space constitute a hybrid set of differential equations,[1] in the sense that the equations for the translation and rotation of the body as a whole are ordinary differential equations and the equations for the elastic motions are partial differential equations. The equations can be derived by the ordinary Lagrangian approach[2] or by Lagrange's equations in terms of quasicoordinates.[1] The latter have the advantage that they are in terms of components along the system body axes, which are more suitable for the design of feedback control.

Lagrange's equations in terms of quasicoordinates are augmented in Ref. 1 by appropriate kinematical relations to obtain a set of hybrid state equations of motion. The state consists of rigid-body displacements in terms of inertial components and velocities in terms of body axes components. The elastic motions are all in terms of body axes components.

This paper presents an alternative approach to that of Ref. 1 in the sense that the state uses quasimomenta instead of quasicoordinates. The equations in terms of quasimomenta are based on the equations in terms of quasicoordinates but they are simpler in form and lend themselves more readily to integration. The hybrid state equations in terms of quasimomenta are then cast in a form suitable for the task of designing controls for the maneuvers of the flexible bodies while controlling the vibration. To this end, a perturbation approach is used to divide the problem into one of design of a high-authority control for the rigid-body maneuver of the body and a low-authority control for the vibration suppression in a manner akin to that used in Ref. 3. For practical reasons, the hybrid perturbation equations for the low-authority control are discretized in space, resulting in a set of first-order differential equations with time-varying coefficients.

II. Hybrid State Equations in Terms of Quasimomenta

We are concerned with the derivation of the equations of motion for a flexible body in space. The motion can be described conveniently by attaching a set of body axes xyz to the body in the undeformed state, where the origin of axes xyz coincides with point 0 (Fig. 1). Then, the motion of the body can be defined by the translation of point 0, the rotation of the axes xyz relative to the inertial axes XYZ and the elastic displacements relative to the body axes xyz. The position of the origin 0 relative to the inertial space is defined by the radius vector $\boldsymbol{R}$, the orientation of the body axes xyz relative to the inertial axes XYZ is defined by the angles θ_i ($i = 1, 2, 3$) and the elastic displacement vector of a typical point in the body relative to axes xyz is denoted by $\boldsymbol{u}$. The angles θ_i can be regarded as the components of a vector $\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is an actual vector when θ_i are small and only a symbolic vector when θ_i are large. Hybrid state equations for a flexible body were derived in Ref. 1. The equations derived in Ref. 1 were in terms of quasicoordinates. As an alternative, in this paper we consider equations in terms of momenta associated with these quasicoordinates. We shall refer to these as *quasimomenta*. From Ref. 1, we can write the hybrid state equations in terms of quasicoordinates in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \boldsymbol{V}}\right) + \tilde{\omega}\frac{\partial L}{\partial \boldsymbol{V}} - C\frac{\partial L}{\partial \boldsymbol{R}} = \boldsymbol{F}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \omega}\right) + \tilde{V}\frac{\partial L}{\partial \boldsymbol{V}} + \tilde{\omega}\frac{\partial L}{\partial \omega} - (D^T)^{-1}\frac{\partial L}{\partial \boldsymbol{\theta}} = \boldsymbol{M} \tag{1}$$
$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{T}}{\partial \boldsymbol{v}}\right) - \frac{\partial \hat{T}}{\partial \boldsymbol{u}} + \mathcal{L}\boldsymbol{u} = \hat{\boldsymbol{f}}$$

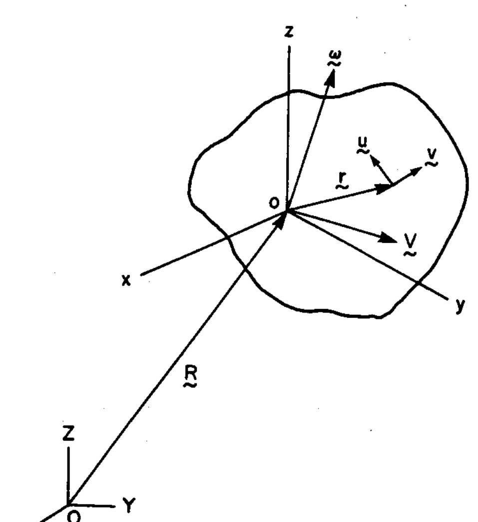

Fig. 1 Flexible body in space.

Presented as Paper 91-2830 at the AIAA Guidance, Navigation, and Control Conference, New Orleans, LA, Aug. 12–14, 1991; received Feb. 4, 1992; revision received Nov. 19, 1992; accepted for publication Nov. 20, 1992.

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where

$$L = T - V \tag{2}$$

is the Lagrangian, in which T is the kinetic energy and V the potential energy, $\boldsymbol{V}$ the velocity vector of point 0, $\boldsymbol{\omega}$ the angular velocity vector of axes xyz, and $\boldsymbol{v}$ is the elastic velocity vector. The symbols $\tilde{V}$ and $\tilde{\omega}$ denote skew symmetric matrices corresponding to the vectors $\boldsymbol{V}$ and $\boldsymbol{\omega}$, respectively; C is a rotation matrix and D is a transformation matrix, both depending on the angles θ_i $(i = 1, 2, 3)$ (Ref. 1). Moreover, $\boldsymbol{F}$ is the resultant force vector, $\boldsymbol{M}$ is the resultant moment vector about 0, and $\hat{\boldsymbol{f}}$ is the force density vector at a typical point in the body. Finally, $\hat{T}$ is the kinetic energy density and $\mathcal{L}$ a stiffness differential operator matrix. The system of equations is hybrid in the sense that the first two of Eqs. (1) are ordinary differential equations and the third is a partial differential equation; the latter is subject to boundary conditions. The operator $\mathcal{L}$ is related to the potential energy by

$$V = \frac{1}{2}[\boldsymbol{u},\boldsymbol{u}] = \int_D \boldsymbol{u}^T \mathcal{L} \boldsymbol{u} \, \mathrm{d}D \tag{3}$$

where [,] is an energy integral[4] and D is the domain of the body. Note that the integral on the right side is obtained from the energy integral through integrations by parts with due consideration to the boundary conditions.

From Ref. 1, the kinetic energy has the expression

$$T = \frac{1}{2} m \boldsymbol{V}^T \boldsymbol{V} + \boldsymbol{V}^T \tilde{S}^T \boldsymbol{\omega} + \boldsymbol{V}^T \int \rho \boldsymbol{v} \, \mathrm{d}D + \boldsymbol{\omega}^T \int \rho (\tilde{r} + \tilde{u}) \boldsymbol{v} \, \mathrm{d}D$$
$$+ \frac{1}{2} \boldsymbol{\omega}^T J \boldsymbol{\omega} + \frac{1}{2} \int \rho \boldsymbol{v}^T \boldsymbol{v} \, \mathrm{d}D \tag{4}$$

where ρ is the mass density and

$$\tilde{S} = \int_D \rho(\tilde{r} + \tilde{u}) \, \mathrm{d}D \tag{5a}$$

$$J = \int_D \rho(\tilde{r} + \tilde{u})(\tilde{r} + \tilde{u})^T \, \mathrm{d}D \tag{5b}$$

Similarly, the kinetic energy density has the form

$$\hat{T} = \tfrac{1}{2} \rho \boldsymbol{V}^T \boldsymbol{V} + \rho \boldsymbol{V}^T (\tilde{r} + \tilde{u})^T \boldsymbol{\omega} + \rho \boldsymbol{V}^T \boldsymbol{v} + \rho \boldsymbol{\omega}^T (\tilde{r} + \tilde{u}) \boldsymbol{v}$$
$$+ \tfrac{1}{2} \rho \boldsymbol{\omega}^T (\tilde{r} + \tilde{u})(\tilde{r} + \tilde{u})^T \boldsymbol{\omega} + \tfrac{1}{2} \rho \boldsymbol{v}^T \boldsymbol{v}$$
$$= \tfrac{1}{2} \rho \boldsymbol{V}^T \boldsymbol{V} + \rho (\boldsymbol{r} + \boldsymbol{u})^T \tilde{\omega}^T \boldsymbol{V} + \rho \boldsymbol{v}^T \boldsymbol{V} + \rho (\boldsymbol{r} + \boldsymbol{u})^T \tilde{\omega}^T \boldsymbol{v}$$
$$+ \tfrac{1}{2} \rho (\boldsymbol{r} + \boldsymbol{u})^T \tilde{\omega}^T \tilde{\omega} (\boldsymbol{r} + \boldsymbol{u}) + \tfrac{1}{2} \rho \boldsymbol{v}^T \boldsymbol{v} \tag{6}$$

Next, we define the quasimomenta as

$$\boldsymbol{p}_V = \frac{\partial L}{\partial \boldsymbol{V}} = m\boldsymbol{V} + \tilde{S}^T \boldsymbol{\omega} + \int \rho \boldsymbol{v} \, \mathrm{d}D$$
$$\boldsymbol{p}_\omega = \frac{\partial L}{\partial \boldsymbol{\omega}} = \tilde{S}\boldsymbol{V} + J\boldsymbol{\omega} + \int \rho (\tilde{r} + \tilde{u}) \boldsymbol{v} \, \mathrm{d}D \tag{7}$$
$$\hat{\boldsymbol{p}}_v = \frac{\partial \hat{T}}{\partial \boldsymbol{v}} = \rho \boldsymbol{V} + \rho (\tilde{r} + \tilde{u})^T \boldsymbol{\omega} + \rho \boldsymbol{v}$$

where $\hat{\boldsymbol{p}}_v$ represents a momentum density vector. The potential energy is assumed to be due entirely to elasticity effects, so that the Lagrangian does not depend on $\boldsymbol{R}$ and $\boldsymbol{\theta}$. Hence,

$$\frac{\partial L}{\partial \boldsymbol{R}} = \boldsymbol{0} \tag{8a}$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}} = \boldsymbol{0} \tag{8b}$$

In the case of space structures, the assumption implies that the gravitational forces and torques are negligibly small compared to the torques and forces required for maneuvering, which is fully justified. Inserting Eqs. (7) and (8) into Eqs. (1), we obtain the hybrid state equations in terms of quasimomenta

$$\dot{\boldsymbol{p}}_V = -\tilde{\omega} \boldsymbol{p}_V + \boldsymbol{F}$$
$$\dot{\boldsymbol{p}}_\omega = -\tilde{V} \boldsymbol{p}_V - \tilde{\omega} \boldsymbol{p}_\omega + \boldsymbol{M} \tag{9}$$
$$\dot{\hat{\boldsymbol{p}}}_v = \hat{\boldsymbol{f}}_u - \mathcal{L}\boldsymbol{u} + \hat{\boldsymbol{f}}$$

in which

$$\hat{\boldsymbol{f}}_u = \frac{\partial \hat{T}}{\partial \boldsymbol{u}} = \rho \tilde{\omega}^T \boldsymbol{V} + \rho \tilde{\omega}^T \boldsymbol{v} + \rho \tilde{\omega}^T \tilde{\omega} (\boldsymbol{r} + \boldsymbol{u}) \tag{10}$$

Equations (9) must be augmented by the kinematical relations

$$\dot{\boldsymbol{R}} = C^T \boldsymbol{V}, \qquad \dot{\boldsymbol{\theta}} = D^{-1} \boldsymbol{\omega}, \qquad \dot{\boldsymbol{u}} = \boldsymbol{v} \tag{11}$$

The state equations, Eqs. (9) and (11), contain the rigid-body velocity vectors $\boldsymbol{V}$ and $\boldsymbol{\omega}$ and the elastic velocity vector $\boldsymbol{v}$. The object is to derive state equations in terms of displacements and momenta alone, so that explicit dependence on velocities must be eliminated. To this end, we first write the momentum equations, Eqs. (8), in the compact symbolic form

$$\boldsymbol{p} = M\boldsymbol{w} \tag{12}$$

where

$$\boldsymbol{p} = \left[\boldsymbol{p}_V^T \;\; \boldsymbol{p}_\omega^T \;\; \hat{\boldsymbol{p}}_v^T \right]^T \tag{13a}$$

$$\boldsymbol{w} = \left[\boldsymbol{V}^T \;\; \boldsymbol{\omega}^T \;\; \boldsymbol{v}^T \right]^T \tag{13b}$$

$$M = \begin{bmatrix} mI & \tilde{S}^T & \int_D \rho \cdots \mathrm{d}D \\ \tilde{S} & J & \int_D \rho(\tilde{r} + \tilde{u}) \cdots \mathrm{d}D \\ \rho I & \rho(\tilde{r} + \tilde{u})^T & \rho I \end{bmatrix} \tag{13c}$$

Moreover, the kinematical relations, Eqs. (11), can be written in the form

$$\dot{\boldsymbol{d}} = T\boldsymbol{w} \tag{14}$$

where

$$\boldsymbol{d} = \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{\theta} \\ \boldsymbol{u} \end{bmatrix} \tag{15a}$$

$$T = \begin{bmatrix} C^T & 0 & 0 \\ 0 & D^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \tag{15b}$$

and the equations of motion, Eqs. (9), in the form

$$\dot{\boldsymbol{p}} = \kappa \boldsymbol{d} + P\boldsymbol{w} + \boldsymbol{P} = \kappa \boldsymbol{d} + PM^{-1}\boldsymbol{p} + \boldsymbol{P} \tag{16}$$

where

$$\kappa = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \tilde{\omega}^T \tilde{\omega} - \mathcal{L} \end{bmatrix} \tag{17a}$$

$$P = \begin{bmatrix} 0 & \tilde{p}_V & 0 \\ \tilde{p}_V & \tilde{p}_\omega & 0 \\ \rho\tilde{\omega}^T & \rho\tilde{\omega}^T\tilde{r}^T & \rho\tilde{\omega}^T \end{bmatrix} \tag{17b}$$

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{M} \\ \hat{\boldsymbol{f}} \end{bmatrix} \tag{17c}$$

where ω, from which $\tilde{\omega}$ is formed, is to be regarded as a function of $\boldsymbol{p}_V$, $\boldsymbol{p}_\omega$, and $\hat{\boldsymbol{p}}_v$ by virtue of Eq. (12). Then, the state equations can be written as

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{X} \tag{18}$$

where

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{d}^T & \boldsymbol{p}^T \end{bmatrix}^T \tag{19a}$$

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{0}^T & \boldsymbol{P}^T \end{bmatrix}^T \tag{19b}$$

are the state vector and the corresponding control vector and

$$A = \begin{bmatrix} 0 & TM^{-1} \\ \kappa & PM^{-1} \end{bmatrix} \tag{20}$$

is the coefficient matrix. Equation (18) is nonlinear, as the coefficient matrix A depends on the state $\boldsymbol{x}$. Moreover, it involves various symbolic operations, as can be concluded from Eqs. (12), (13), (16), and (17).

III. Perturbation Technique for Maneuvering and Control

The state equations, Eq. (18), are quite general in the sense that they can be used to treat any problem involving the dynamics and control of a flexible body. One problem of particular interest is the maneuvering of the body. Ideally, the maneuver should be carried out without exciting elastic vibration, i.e., as if the body were rigid. In reality, the body is not rigid, so that some elastic vibration will be excited. In most practical cases, the elastic vibration tends to be small compared to the motions characterizing the rigid-body maneuvering. Hence, one can conceive of a dual-level control: the first, a high-authority control, designed to control the maneuvering of the body as if it were rigid, and the second, a low-authority control, designed to control the elastic vibration and any deviations from the rigid-body maneuvering. Because the quantities involved in the two control levels differ in the magnitude order, it is only natural to adopt a perturbation approach whereby the high-authority control is associated with a zero-order problem and the low-authority control with a first-order problem, both "zero" and "first" referring to the magnitude order in the perturbation theory sense.[2] This approach permits solving the high-authority control problem independently of the low-authority control. On the other hand, the low-authority control problem formulation will contain time-varying terms and persistent disturbances depending on the maneuvering motions characterizing the high-authority control problem.

The perturbation approach implies that the quantities entering into the equations of motion can be expressed as follows:

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{x}_1, \qquad \boldsymbol{X} = \boldsymbol{X}_0 + \boldsymbol{X}_1, \qquad A = A_0 + A_1 \tag{21}$$

where the subscript zero indicates quantities of one order of magnitude larger than quantities with the subscript one. We note that

$$\boldsymbol{x}_0 = \begin{bmatrix} \boldsymbol{R}_0^T & \boldsymbol{\theta}_0^T & \boldsymbol{0}^T & \boldsymbol{p}_{V0}^T & \boldsymbol{p}_{\omega 0}^T & \hat{\boldsymbol{p}}_{v0}^T \end{bmatrix}^T$$
$$\boldsymbol{x}_1 = \begin{bmatrix} \boldsymbol{R}_1^T & \boldsymbol{\theta}_1^T & \boldsymbol{u} & \boldsymbol{p}_{V1}^T & \boldsymbol{p}_{\omega 1}^T & \hat{\boldsymbol{p}}_{v1}^T \end{bmatrix}^T \tag{22}$$

which takes into account that $\boldsymbol{u}_0 \equiv \boldsymbol{0}$ and $\boldsymbol{u}_1 \equiv \boldsymbol{u}$, consistent with the assumption that elastic motions are small. Inserting Eqs. (21) into Eq. (18) and separating terms of different orders of magnitude, while ignoring second-order terms, we obtain

$$\dot{\boldsymbol{x}}_0 = A_0\boldsymbol{x}_0 + \boldsymbol{X}_0 \tag{23a}$$

$$\dot{\boldsymbol{x}}_1 = A_0\boldsymbol{x}_1 + A_1\boldsymbol{x}_0 + \boldsymbol{X}_1 \tag{23b}$$

Equation (23a) describes the zero-order problem associated with the high-authority control and Eq. (23b) the first-order problem associated with the low-authority control.

The problem of determining the matrices A_0 and A_1 remains. To carry out this task, it is convenient to return to Eqs. (14) and (16). Inserting

$$\boldsymbol{d} = \boldsymbol{d}_0 + \boldsymbol{d}_1 \tag{24a}$$

$$T = T_0 + T_1 \tag{24b}$$

$$\boldsymbol{w} = \boldsymbol{w}_0 + \boldsymbol{w}_1 \tag{24c}$$

into Eq. (14) and following the procedure used earlier, we obtain

$$\dot{\boldsymbol{d}}_0 = T_0\boldsymbol{w}_0 \tag{25a}$$

$$\dot{\boldsymbol{d}}_1 = T_0\boldsymbol{w}_1 + T_1\boldsymbol{w}_0 \tag{25b}$$

Moreover, inserting

$$\boldsymbol{p} = \boldsymbol{p}_0 + \boldsymbol{p}_1 \tag{26a}$$

$$M = M_0 + M_1 \tag{26b}$$

together with Eq. (24c) into Eq. (12), we have

$$\boldsymbol{p}_0 = M_0\boldsymbol{w}_0 \tag{27a}$$

$$\boldsymbol{p}_1 = M_0\boldsymbol{w}_1 + M_1\boldsymbol{w}_0 \tag{27b}$$

so that Eqs. (25) can be rewritten as follows:

$$\dot{\boldsymbol{d}}_0 = T_0 M_0^{-1}\boldsymbol{p}_0 \tag{28a}$$

$$\dot{\boldsymbol{d}}_1 = T_0 M_0^{-1}\boldsymbol{p}_1 + (T_1 - T_0 M_0^{-1} M_1)\boldsymbol{w}_0 \tag{28b}$$

Next, we insert

$$\kappa = \kappa_0 + \kappa_1 \tag{29a}$$

$$P = P_0 + P_1 \tag{29b}$$

$$\boldsymbol{P} = \boldsymbol{P}_0 + \boldsymbol{P}_1 \tag{29c}$$

together with Eqs. (24a), (24c), and (26a) into Eq. (16) and follow the established procedure to obtain

$$\dot{\boldsymbol{p}}_0 = \kappa_0\boldsymbol{d}_0 + P_0\boldsymbol{w}_0 + \boldsymbol{P}_0 \tag{30a}$$

$$\dot{\boldsymbol{p}}_1 = \kappa_0\boldsymbol{d}_1 + P_0\boldsymbol{w}_1 + \kappa_1\boldsymbol{d}_0 + P_1\boldsymbol{w}_0 + \boldsymbol{P}_1 \tag{30b}$$

so that, considering Eqs. (27) and recognizing the $\kappa_0\boldsymbol{d}_0 = \boldsymbol{0}$ and $\kappa_1\boldsymbol{d}_0 = \boldsymbol{0}$, because $\kappa\boldsymbol{d}_0 = \boldsymbol{0}$, Eqs. (30) can be rewritten in the form

$$\dot{\boldsymbol{p}}_0 = P_0 M_0^{-1}\boldsymbol{p}_0 + \boldsymbol{P}_0 \tag{31a}$$

$$\dot{\boldsymbol{p}}_1 = \kappa_0\boldsymbol{d}_1 + P_0 M_0^{-1}\boldsymbol{p}_1 + (P_1 - P_0 M_0^{-1} M_1)\boldsymbol{w}_0 + \boldsymbol{P}_1 \tag{31b}$$

A. Equations for High-Authority Control

The high-authority control is charged with the rigid-body maneuvering of the body and must satisfy Eq. (23a) on the one hand and Eqs. (28a) and (31a) on the other, from which we conclude that the coefficient matrix in Eq. (23a) has the form

$$A_0 = \begin{bmatrix} 0 & T_0 M_0^{-1} \\ 0 & P_0 M_0^{-1} \end{bmatrix} \tag{32}$$

where

$$T_0 = \begin{bmatrix} C_0^T & 0 & 0 \\ 0 & D_0^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \tag{33a}$$

$$M_0 = \begin{bmatrix} mI & \tilde{S}_0^T & \int_D \rho \cdots \mathrm{d}D \\ \tilde{S}_0 & J_0 & \int_D \rho \tilde{r} \cdots \mathrm{d}D \\ \rho I & \rho \tilde{r}^T & \rho I \end{bmatrix} \tag{33b}$$

$$P_0 = \begin{bmatrix} 0 & \tilde{p}_{V0} & 0 \\ \tilde{p}_{V0} & \tilde{p}_{\omega 0} & 0 \\ \rho\omega_0^T & \rho\tilde{\omega}_0^T \tilde{r}^T & \rho\tilde{\omega}_0^T \end{bmatrix} \tag{33c}$$

in which

$$\tilde{S}_0 = \int_D \rho \tilde{r} \, \mathrm{d}D \tag{34a}$$

$$J_0 = \int_D \rho \tilde{r} \tilde{r}^T \mathrm{d}D \tag{34b}$$

Because the elastic motion is absent from the zero-order equation, we can rewrite Eq. (23a) in the reduced form

$$\dot{x}_0^* = A_0^* x_0^* + X_0^* \tag{35}$$

where

$$x_0^* = \left[R_0^T \quad \theta_0^T \quad p_{V0}^T \quad p_{\omega 0}^T \right]^T \tag{36a}$$

$$X_0^* = \left[0^T \quad 0^T \quad F_0^T \quad M_0^T \right]^T \tag{36b}$$

$$A_0^* = \begin{bmatrix} 0 & T_0^*(M_0^*)^{-1} \\ 0 & P_0^*(M_0^*)^{-1} \end{bmatrix} \tag{36c}$$

are vectors and matrix of reduced dimensions, in which

$$T_0^* = \begin{bmatrix} C_0^T & 0 \\ 0 & D_0^{-1} \end{bmatrix} \tag{37a}$$

$$M_0^* = \begin{bmatrix} mI & \tilde{S}_0^T \\ \tilde{S}_0 & J_0 \end{bmatrix} \tag{37b}$$

$$P_0^* = \begin{bmatrix} 0 & \tilde{p}_{V0} \\ \tilde{p}_{V0} & \tilde{p}_{\omega 0} \end{bmatrix} \tag{37c}$$

Moreover, from the equation for the elastic motion, given by the bottom vector equation in the state equations, Eq. (18), we can write the distributed force required for rigid-body maneuvering

$$\hat{f}_0 = \dot{\hat{p}}_{v0} - \hat{f}_{u0} = \rho(\dot{V}_0 + \tilde{r}^T \dot{\omega}_0 - \tilde{\omega}_0^T V_0 - \tilde{\omega}_0^T \tilde{\omega}_0 r) \tag{38}$$

The absence of such a force is equivalent to a body force equal to $-\hat{f}_0$ acting on the system.

Equation (34) can be used to design controls for the rigid-body maneuvering. This problem has been investigated earlier[5] and is not pursued here.

B. Perturbation Equations for the Low-Authority Control

The perturbation equations were essentially derived earlier in this section in the form of Eqs. (28b) and (31b), but many details remain to be worked out. At this point, we turn our attention to this task.

In the first place, we note that it is only proper that Eqs. (28b) and (31b) contain w_0 rather than p_0, because as far as the first-order problem is concerned w_0 is known. From Eq. (15a), we conclude that

$$T_1 = \sum_{i=1}^{3} \left.\frac{\partial T}{\partial \theta_i}\right|_{\theta_{i0}} \theta_{i1} = \sum_{i=1}^{3} \begin{bmatrix} \left.\frac{\partial C^T}{\partial \theta_i}\right|_{\theta_{i0}} & 0 & 0 \\ 0 & \left.\frac{\partial D^{-1}}{\partial \theta_i}\right|_{\theta_{i0}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \theta_{i1} \tag{39}$$

so that

$$T_1 w_0 = \sum_{i=1}^{3} \begin{bmatrix} \left.\frac{\partial C^T}{\partial \theta_i}\right|_{\theta_{i0}} V_0 \theta_{i1} \\ \left.\frac{\partial D^{-1}}{\partial \theta_i}\right|_{\theta_{i0}} \omega_0 \theta_{i1} \\ 0 \end{bmatrix} = \Theta_0 \theta_1 = [0 \quad \Theta_0 \quad 0] d_1 \tag{40}$$

where

$$\Theta_0 = \begin{bmatrix} V_0 \\ \Omega_0 \\ 0 \end{bmatrix} \tag{41}$$

in which

$$V_0 = \left[\left.\frac{\partial C^T}{\partial \theta_1}\right|_{\theta_{i0}} V_0 \quad \left.\frac{\partial C^T}{\partial \theta_2}\right|_{\theta_{i0}} V_0 \quad \left.\frac{\partial C^T}{\partial \theta_3}\right|_{\theta_{i0}} V_0 \right] \tag{42a}$$

$$\Omega_0 = \left[\left.\frac{\partial D^{-1}}{\partial \theta_1}\right|_{\theta_{i0}} \omega_0 \quad \left.\frac{\partial D^{-1}}{\partial \theta_2}\right|_{\theta_{i0}} \omega_0 \quad \left.\frac{\partial D^{-1}}{\partial \theta_3}\right|_{\theta_{i0}} \omega_0 \right] \tag{42b}$$

Moreover,

$$M_1 = \begin{bmatrix} 0 & \int_D \rho \tilde{u}^T \mathrm{d}D & 0 \\ \int_D \rho \tilde{u}^T \mathrm{d}D & \int_D \rho(\tilde{r}\tilde{u}^T + \tilde{u}\tilde{r}^T) \, \mathrm{d}D & \int_D \rho \tilde{u}^T \cdots \mathrm{d}D \\ 0 & \rho \tilde{u}^T & 0 \end{bmatrix} \tag{43a}$$

$$P_1 = \begin{bmatrix} 0 & \tilde{p}_{V1} & 0 \\ \tilde{p}_{V1} & \tilde{p}_{\omega 1} & 0 \\ \rho\tilde{\omega}_1^T & \rho\tilde{\omega}_1^T \tilde{r}^T & \rho\tilde{\omega}_1^T \end{bmatrix} \tag{43b}$$

so that, using Eq. (43a),

$$M_1 w_0 = \begin{bmatrix} \int_D \rho \tilde{u}^T \mathrm{d}D \omega_0 \\ \int_D \rho \tilde{u} \, \mathrm{d}D V_0 + \int_D \rho(\tilde{r}\tilde{u}^T + \tilde{u}\tilde{r}^T) \, \mathrm{d}D \omega_0 \\ \rho \tilde{u}^T \omega_0 \end{bmatrix}$$

$$= U_0 u = [0 \quad 0 \quad U_0] d_1 \tag{44}$$

where

$$U_0 = \begin{bmatrix} \tilde{\omega}_0 \int_D \rho \cdots \mathrm{d}D \\ -\int_D \rho\left(\tilde{V}_0 - \tilde{r}\tilde{\omega}_0 + \widetilde{\tilde{\omega}_0 r}\right) \cdots \mathrm{d}D \\ \rho\tilde{\omega}_0 \end{bmatrix} \tag{45}$$

Inserting Eqs. (40) and (44) into Eq. (28b), we obtain

$$\dot{d}_1 = \mathfrak{D}_0 d_1 + T_0 M_0^{-1} p_1 \tag{46}$$

where

$$\mathfrak{D}_0 = \begin{bmatrix} 0 & \Theta_0 & -T_0 M^{-1} U_0 \end{bmatrix} \tag{47}$$

Moreover, inserting Eq. (44) into Eq. (27b), we conclude that

$$p_1 = M_0 w_1 + U_0 u \tag{48}$$

In addition, using Eq. (43b),

$$P_1 w_0 = \begin{bmatrix} 0 & \tilde{p}_{V1} & 0 \\ \tilde{p}_{V1} & \tilde{p}_{\omega 1} & 0 \\ \rho\tilde{\omega}_1^T & \rho\tilde{\omega}_1^T \tilde{r}^T & \rho\tilde{\omega}_1^T \end{bmatrix} \begin{bmatrix} V_0 \\ \omega_0 \\ \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{p}_{V1}\omega_0 \\ \tilde{p}_{V1} V_0 + \tilde{p}_{\omega 1}\omega_0 \\ \rho\tilde{\omega}_1^T (V_0 + \tilde{r}^T \omega_0) \end{bmatrix} = \begin{bmatrix} -\tilde{\omega}_0 p_{V1} \\ -\tilde{V}_0 p_{V1} - \tilde{\omega}_0 p_{\omega 1} \\ \rho\left(\tilde{V}_0 + \widetilde{\tilde{r}^T \omega_0}\right)\omega_1 \end{bmatrix} \tag{49}$$

But, from Eq. (48),

$$\omega_1 = \left[M_0^{-1}\right]_{\omega 1} (p_1 - U_0 u) \tag{50}$$

where $[M_0^{-1}]_{\omega 1}$ is a submatrix of M_0^{-1} corresponding to ω_1, so that

$$P_1 w_0 = \begin{bmatrix} 0 \\ 0 \\ -\rho\left(\tilde{V}_0 + \widetilde{\tilde{r}^T \omega_1}\right)\left[M_0^{-1}\right]_{\omega 1} U_0 \end{bmatrix} u - \begin{bmatrix} \tilde{\omega}_0 & 0 & 0 \\ \tilde{V}_0 & \tilde{\omega}_0 & 0 \\ \hline \multicolumn{3}{c}{-\rho\left(\tilde{V}_0 + \widetilde{\tilde{r}^T \omega_0}\right)\left[M_0^{-1}\right]_{\omega 1}} \end{bmatrix} p_1$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\rho\left(\tilde{V}_0 + \widetilde{\tilde{r}^T \omega_0}\right)\left[M_0^{-1}\right]_{\omega 1} U_0 \end{bmatrix} d_1 - \begin{bmatrix} \tilde{\omega}_0 & 0 & 0 \\ \tilde{V}_0 & \tilde{\omega}_0 & 0 \\ \hline \multicolumn{3}{c}{-\rho\left(\tilde{V}_0 + \widetilde{\tilde{r}^T \omega_0}\right)\left[M_0^{-1}\right]_{\omega 1}} \end{bmatrix} p_1 \tag{51}$$

Hence, inserting Eqs. (44) and (51) into Eq. (31b), we can write

$$\dot{p}_1 = (\kappa_0 - \mathfrak{F}_0) d_1 + \left(P_0 M_0^{-1} - \mathcal{E}_0\right) p_1 + P_1 \tag{52}$$

in which

$$\kappa_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho\tilde{\omega}_0^T \tilde{\omega}_0 - \mathfrak{L} \end{bmatrix} \tag{53a}$$

$$\mathfrak{F}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho\left(\tilde{V}_0 + \widetilde{\tilde{r}^T \omega_0}\right)\left[M_0^{-1}\right]_{\omega 1} U_0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & P_0 M_0^{-1} U_0 \end{bmatrix} \tag{53b}$$

$$\mathcal{E}_0 = \begin{bmatrix} \tilde{\omega}_0 & 0 & 0 \\ \tilde{V}_0 & \tilde{\omega}_0 & 0 \\ \hline \multicolumn{3}{c}{-\rho\left(\tilde{V}_0 + \widetilde{\tilde{r}^T \omega_0}\right)\left[M_0^{-1}\right]_{\omega 1}} \end{bmatrix} \tag{53c}$$

Equations (46) and (52) can be combined into

$$\dot{x}_1 = A_p x_1 + X_1 \tag{54}$$

where

$$A_p = \begin{bmatrix} \mathfrak{D}_0 & T_0 M_0^{-1} \\ \kappa_0 - \mathfrak{F}_0 & P_0 M_0^{-1} - \mathcal{E}_0 \end{bmatrix} \tag{55}$$

is a time-varying coefficient matrix for the perturbation equations. Unlike Eq. (35), however, Eq. (54) is linear, albeit with time-varying coefficients.

IV. Spatial Discretization of the Perturbation Equations

The hybrid perturbation equations involve certain symbolic operations likely to cause serious difficulties. Almost invariably, the difficulties are caused by the partial differential equations. To circumvent these difficulties, it is necessary to transform the partial differential equations into sets of ordinary differential equations, which implies spatial discretization of the distributed variables.

Let us express the elastic displacement and velocity vectors in the form

$$u(x,y,z,t) = \Phi(x,y,z)\xi(t) \tag{56a}$$

$$v(x,y,z,t) = \Phi(x,y,z,t)\eta(t) \tag{56b}$$

$$\dot{\xi}(t) = \eta(t) \tag{56c}$$

where $\Phi(x,y,z)$ is a matrix of admissible functions[4] and $\xi(t)$ and $\eta(t)$ are vectors of generalized displacements and velocities, respectively. We assume that the admissible functions have been normalized so as to satisfy

$$\int_D \rho\Phi^T\Phi \,\mathrm{d}D = I \tag{57}$$

The spatial discretization involves premultiplication of the equation for the distributed momentum vector $\hat{p}_{v1}$ and of the partial differential vector equation associated with $\hat{p}_{v1}$ by Φ^T and integration of the resulting equations over the elastic domain D. In the process, we denote the discretized momentum vector associated with the elastic motions by

$$p_\eta = \int_D \Phi^T \hat{p}_{v1} \,\mathrm{d}D \tag{58}$$

Then, it can be verified that the perturbed momentum vector, Eq. (48), can be written in the form

$$p_1 = M_0 w_1 + U_0 \xi \tag{59}$$

where this time

$$p_1 = \begin{bmatrix} p_{V1}^T & P_{\omega 1}^T & p_\eta^T \end{bmatrix} \tag{60a}$$

$$w_1 = \begin{bmatrix} V_1^T & \omega_1^T & \eta^T \end{bmatrix}^T \tag{60b}$$

$$M_0 = \begin{bmatrix} mI & \tilde{S}_0^T & \bar{\Phi} \\ \tilde{S}_0 & J_0 & \tilde{\bar{\Phi}} \\ \bar{\Phi}^T & \tilde{\bar{\Phi}}^T & I \end{bmatrix} \tag{60c}$$

in which

$$\bar{\Phi} = \int_D \rho \Phi \, dD \tag{61a}$$

$$\tilde{\bar{\Phi}} = \int_D \rho \tilde{r} \Phi \, dD \tag{61b}$$

and

$$U_0 = \begin{bmatrix} \tilde{\omega}_0 \bar{\Phi} \\ -H_1 \\ H_0 \end{bmatrix} \tag{62}$$

where

$$H_0 = H_0(\omega_0) = \int_D \rho \Phi^T \tilde{\omega}_0 \Phi \, dD \tag{63a}$$

$$H_1 = H_1(V_0, \omega_0) = \int_D \rho \left(\tilde{V}_0 - \tilde{r} \tilde{\omega}_0 + \widetilde{\tilde{\omega}_0 r} \right) \Phi \, dD \tag{63b}$$

At this point, we are in the position to put together the perturbation state equations. It is easy to verify that Eq. (52) retains its form, except that x_1 and X_1, are defined now as

$$x_1 = \begin{bmatrix} R_1^T & \theta_1^T & \xi^T & p_{V1}^T & p_{\omega 1}^T & p_\eta^T \end{bmatrix}^T \tag{64a}$$

$$X_1 = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & F_1^T & M_1^T & f^T \end{bmatrix}^T \tag{64b}$$

in which

$$f = \int_D \Phi^T \hat{f} \, dD \tag{65}$$

Moreover, the various matrices appearing in the coefficient matrix A_p are as follows:

$$\kappa_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & H_2 - K \end{bmatrix} \tag{66a}$$

$$\mathfrak{F}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & H_3 \left[M_0^{-1} \right]_{\omega 1} U_0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & P_0 M_0^{-1} U_0 \end{bmatrix} \tag{66b}$$

$$P_0 = \begin{bmatrix} 0 & \tilde{p}_{V0} & 0 \\ \tilde{p}_{V0} & \tilde{p}_{\omega 0} & 0 \\ \bar{\Phi}^T \tilde{\omega}_0^T & H_4 & H_0 \end{bmatrix} \tag{66c}$$

$$\mathcal{E}_0 = \begin{bmatrix} \tilde{\omega}_0 & 0 & 0 \\ \tilde{V}_0 & \tilde{\omega}_0 & 0 \\ \hline & -H_3 [M_0^{-1}]_{\omega 1} & \end{bmatrix} \tag{66d}$$

in which

$$K = \int_D \Phi^T \mathcal{L} \Phi \, dD \tag{67a}$$

$$H_2 = H_2(\omega_0) = \int_D \rho \Phi^T \tilde{\omega}_0^T \tilde{\omega}_0 \Phi \, dD \tag{67b}$$

$$H_3 = H_3(V_0, \omega_0) = \int_D \rho \Phi^T (\tilde{V}_0 + \widetilde{\tilde{r}^T \omega_0}) \, dD \tag{67c}$$

$$H_4 = H_4(\omega_0) = \int_D \rho \Phi^T \tilde{\omega}_0^T \tilde{r} \, dD \tag{67d}$$

In addition, we recall that M_0 is given by Eq. (60c).

V. Feedback Control of the Perturbations

The ideal maneuver is that in which the body moves as if it were rigid, i.e., with the perturbed state x_1 identically zero. This is not practical and may not be even feasible. Consequently, the object is to drive the perturbations to zero within a reasonable interval of time. To this end, we propose to use optimal feedback control based on the linear quadratic regulator (LQR) theory. The control design is based on Eq. (54), with X_1 representing the combination of the disturbance vector and the control vector. Before proceeding with the control design, we must relate the vector X_1 to the actuator forces.

Let us assume that the low-authority control is carried out by means of N point actuators exerting forces $u_i(t)$ at points x_i, y_i, z_i throughout the body ($i = 1, 2, \ldots, N$). Discrete forces can be treated as distributed through the use of spatial Dirac delta functions. Hence, let us assume that the force density vector can be written in the form

$$\hat{f}(x,y,z,t) = \hat{f}_d(x,y,z,t) + \sum_{i=1}^{N} u_i(t) \delta(x - x_i, y - y_i, z - z_i) \tag{68}$$

where

$$\hat{f}_d = -\hat{f}_0 = -\rho(\dot{V}_0 + \tilde{r}^T \dot{\omega}_0 - \tilde{\omega}_0^T V_0 - \tilde{\omega}_0^T \tilde{\omega}_0 r) \tag{69}$$

is a disturbance force density vector and $\delta(x - x_i, y - y_i, z - z_i)$ are spatial Dirac delta functions defined as

$$\begin{gathered} \delta(x - x_i, y - y_i, z - z_i) = 0, \quad x \neq x_i, \; y \neq y_i, \; z \neq z_i \\ \int_D \delta(x - x_i, y - y_i, z - z_i) \, dD = 1 \end{gathered} \tag{70}$$

Then, the resultant force on the body is simply

$$\begin{aligned} F_1(t) &= \int_D \hat{f}(x,y,z,t) \, dD = F_{d1}(t) \\ &\quad + \int_D \sum_{i=1}^{N} u_i(t) \delta(x - x_i, y - y_i, z - z_i) \, dD \\ &= F_{d1}(t) + \sum_{i=1}^{N} u_i(t) = F_{d1}(t) + [I \quad I \quad \cdots \quad I] u(t) \end{aligned} \tag{71}$$

where

$$F_{d1}(t) = -\int_D \rho(\dot{V}_0 + \tilde{r}^T \dot{\omega}_0 - \tilde{\omega}^T V_0 - \tilde{\omega}_0^T \tilde{\omega}_0 r) \, dD \tag{72}$$

is a disturbance force vector, I are 3×3 identity matrices and

$$u(t) = \left[u_1^T(t) \quad u_2^T(t) \quad \cdots \quad u_N^T(t)\right]^T \tag{73}$$

is a $3N$ vector. Similarly, the resultant torque on the body has the form

$$\begin{aligned} M_1(t) &= \int_D \tilde{r} f(x,y,z,t)\,\mathrm{d}D = M_{d1}(t) \\ &+ \int_D \tilde{r} \sum_{i=1}^{N} u_i(t)\,\delta(x-x_i,\ y-y_i,\ z-z_i)\,\mathrm{d}D \\ &= M_{d1}(t) + \sum_{i=1}^{N} \tilde{r}_i u_i(t) = M_{d1}(t) \\ &+ \left[\tilde{r}_1 \quad \tilde{r}_2 \quad \cdots \quad \tilde{r}_N\right] u(t) \end{aligned} \tag{74}$$

where

$$M_{d1}(t) = -\int_D \rho \tilde{r}(\dot{V}_0 + \tilde{r}^T\dot{\omega}_0 - \tilde{\omega}_0^T V_0 - \tilde{\omega}_0^T\tilde{\omega}_0 r)\,\mathrm{d}D \tag{75}$$

is a disturbance torque vector. Moreover, as indicated by Eq. (65), the generalized force vector for the elastic motions can be written as

$$\begin{aligned} f(t) &= \int_D \Phi^T(x,y,z) f(x,y,z,t)\,\mathrm{d}D \\ &= f_d(t) + \int_D \Phi^T(x,y,z) \sum_{i=1}^{N} u_i(t) \\ &\times \delta(x-x_i,\ y-y_i,\ z-z_i)\,\mathrm{d}D \\ &= f_d(t) + \sum_{i=1}^{N} \Phi_i^T u_i(t) = f_d(t) \\ &+ [\Phi_1^T \quad \Phi_2^T \quad \ldots \quad \Phi_N^T] u(t) \end{aligned} \tag{76}$$

where

$$f_d(t) = -\int_D \rho \Phi^T(\dot{V}_0 + \tilde{r}^T\dot{\omega}_0 - \tilde{\omega}_0^T V_0 - \tilde{\omega}_0^T\tilde{\omega}_0 r)\,\mathrm{d}D \tag{77}$$

is a generalized disturbance force vector and

$$\Phi_i = \Phi(x_i, y_i, z_i), \qquad i = 1, 2, \ldots, N \tag{78}$$

Equations (71), (74), and (76) can be combined into

$$P_1(t) = \left[F_1^T(t) \quad M_1^T(t) \quad f^T(t)\right]^T = P_{d1}(t) + Bu(t) \tag{79}$$

where

$$P_{d1}(t) = \begin{bmatrix} F_{d1}(t) \\ M_{d1}(t) \\ f_d(t) \end{bmatrix}, \qquad B = \begin{bmatrix} I & I & \cdots & I \\ \tilde{r}_1 & \tilde{r}_2 & \cdots & \tilde{r}_N \\ \Phi_1^T & \Phi_2^T & \cdots & \Phi_N^T \end{bmatrix} \tag{80}$$

Finally, the desired relation between X_1 and the actuator force vector $u(t)$ has the form

$$X_1(t) = \left[0^T \quad P_1^T(t)\right]^T = X_{d1}(t) + B'u(t) \tag{81}$$

where

$$X_{d1}(t) = \begin{bmatrix} 0 \\ P_{d1}(t) \end{bmatrix}, \qquad B' = \begin{bmatrix} 0 \\ B \end{bmatrix} \tag{82}$$

It follows that the state equations for the low-authority control can be written in the standard form

$$\dot{x}_1(t) = A(t)x_1(t) + X_{d1}(t) + B'u(t) \tag{83}$$

where we dropped the subscript p from the coefficient matrix.

Equation (83) represents a time-varying linear system subjected to persistent disturbances. The control design can be carried out by the LQR theory modified so as to accommodate the persistent disturbances, in a manner similar to the approach developed in Ref. 6. To this end, we assume that the desired optimal control minimizes the quadratic performance measure

$$J = \frac{1}{2} x_1^T(t_f) H x_1(t_f) + \frac{1}{2}\int_{t_i}^{t_f} \left[x_1^T(t) Q x_1(t) + u^T(t) R u(t)\right] \mathrm{d}t \tag{84}$$

where H and Q are real symmetric positive semidefinite matrices and R is a real symmetric positive definite matrix; t_i and t_f are the initial and final time, respectively. To derive the optimal control law, we introduce the Hamiltonian[7]

$$\mathcal{H} = \tfrac{1}{2} x_1^T Q x_1 + \tfrac{1}{2} u^T R u + p^T(Ax_1 + B'u + X_{d1}) \tag{85}$$

where p is the costate vector. Then, the necessary conditions for optimality are[7]

$$\dot{x}_1 = \frac{\partial \mathcal{H}}{\partial p} = Ax_1 + B'u + X_{d1} \tag{86a}$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial x_1} = -(Qx_1 + A^T p) \tag{86b}$$

$$0 = \frac{\partial \mathcal{H}}{\partial u} = Ru + (B')^T p \tag{86c}$$

Moreover,

$$p(t_f) = H(t_f) x_1(t_f) \tag{87}$$

From Eq. (86c), we obtain the control law

$$u = -R^{-1}(B')^T p \tag{88}$$

Equation (88) contains the costate vector p, which can be eliminated by assuming that

$$p = Kx_1 + s \tag{89}$$

where K is a matrix and s is a vector, both depending on time and still to be determined. From Eqs. (87) and (89), we conclude that K and s must satisfy the end conditions

$$K(t_f) = H(t_f) \tag{90a}$$

$$s(t_f) = 0 \tag{90b}$$

Then, combining Eqs. (86a), (86b), (88), and (89), we obtain

$$\begin{aligned} \dot{p} = \dot{K}x_1 + K\dot{x}_1 + \dot{s} &= \left[\dot{K} + KA - KB'R^{-1}(B')^T K\right] x_1 \\ &- KB'R^{-1}(B')^T s + KX_{d1} + \dot{s} \\ &= -(Q + A^T K)x_1 - A^T s \end{aligned} \tag{91}$$

To satisfy Eq. (91), as well as Eqs. (90), we choose K and s so that

$$\dot{K} = -Q - KA - A^T K + KB'R^{-1}(B')^T K, \quad K(t_f) = H(t_f) \tag{92}$$

and

$$\dot{s} = \left[KB'R^{-1}(B')^T - A^T\right] s - KX_{d1}, \qquad s(t_f) = 0 \tag{93}$$

Equation (92) represents the transient Riccati equation, a nonlinear matrix differential equation to be integrated backward in time to obtain $K(t)$. Instead of solving a nonlinear matrix differential equation, it is possible to transform the problem into a linear one of twice the order.[7] Having K, Eq. (93) must be solved to obtain $s(t)$. Then, inserting Eq. (89) into Eq. (88), we obtain the optimal control law

$$u = -R^{-1}(B')^T(Kx_1+s) \tag{94}$$

and note that the control given by Eq. (94) is designed to mitigate the effect of both transient and persistent disturbances.

Implementation of the control law, Eq. (94), requires measurements of the state, Eq. (64a), which implies measurements of displacements and momenta, when in fact the sensors measure displacements and velocities. This presents no particular problem in view of the fact the momenta and velocities are related, as indicated by Eq. (59). Introducing the notation

$$U_0' = \begin{bmatrix} 0 \\ 0 \\ U_0 \end{bmatrix} \tag{95}$$

Eq. (59) can be rewritten as

$$p_1 = U_0' d_1 + M_0 w_1 \tag{96}$$

where d_1 is the top-half of the discretized state vector. Then, adjoining the identity $d_1 \equiv d_1$ to Eq. (96), we can write the relation between the measurement vector and the state vector in the form

$$x_1 = N y_1 \tag{97}$$

where

$$y_1 = \begin{bmatrix} d_1 \\ w_1 \end{bmatrix}, \qquad N = \begin{bmatrix} I & 0 \\ U_0' & M_0 \end{bmatrix} \tag{98}$$

Equation (97) represents an output equation and implies that the sensor output vector is of the same dimension as the state vector. If the output vector is of lower dimension, then the state can be estimated by means of a Luenberger observer or a Kalman filter.[7]

VI. Example

As an illustration, we propose to derive the state equations of motion for a system consisting of a rigid cylindrical hub and a flexible appendage, as depicted in Fig. 2; it is the same system

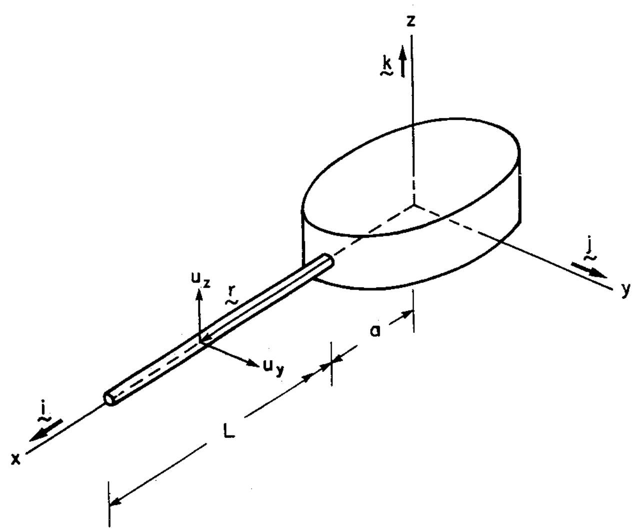

Fig. 2 Rigid body with a flexible appendage.

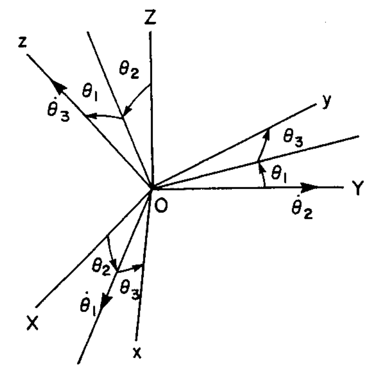

Fig. 3 Angular displacements of the body axes.

as that considered in Ref. 1. From Fig. 2, we can write

$$r = xi, \quad u = u_y j + u_z k, \quad v = v_y j + v_z k, \quad a < x < a + L \tag{99}$$

where i, j, and k are unit vectors. The matrix M for the hybrid system is given by Eq. (13c), in which, according to Eqs. (5),

$$\tilde{S} = \begin{bmatrix} 0 & -\int \rho u_z \, dx & \int \rho u_y \, dx \\ \int \rho u_z \, dx & 0 & -m_1 \bar{x} \\ -\int \rho u_y \, dx & m_1 \bar{x} & 0 \end{bmatrix} \tag{100a}$$

$$J = \begin{bmatrix} J_{xx} + \int \rho(u_y^2 + u_z^2) \, dx & -\int \rho x u_y \, dx & -\int \rho x u_z \, dx \\ -\int \rho x u_y \, dx & J_{yy} + \int \rho u_z^2 \, dx & -\int \rho u_y u_z \, dx \\ -\int \rho x u_z \, dx & -\int \rho u_y u_z \, dx & J_{zz} + \int \rho u_y^2 \, dx \end{bmatrix} \tag{100b}$$

where ρ is the mass density of the appendage, m_1 the total mass of the appendage, $\bar{x}$ the position of the mass center of the appendage; and J_{xx}, J_{yy}, and J_{zz} are the mass moments of inertia of the whole body regarded as rigid. Moreover, as in Ref. 1, we assumed that the orientation of the body axes xyz relative to the inertial axes XYZ is defined by the rotations θ_i $(i = 1, 2, 3)$, as shown in Fig. 3. Hence, from Ref. 1, we obtain

$$C = \begin{bmatrix} c_2c_3 + s_1s_2s_3 & c_1s_3 & -s_2c_3 + s_1c_2s_3 \\ -c_2s_3 + s_1s_2c_3 & c_1c_3 & s_2s_3 + s_1c_2c_3 \\ c_1s_2 & -s_1 & c_1c_2 \end{bmatrix} \tag{101a}$$

$$D = \begin{bmatrix} c_3 & c_1s_3 & 0 \\ s_3 & c_1c_3 & 0 \\ 0 & -s_1 & 1 \end{bmatrix} \tag{101b}$$

in which $s_i = \sin\theta_i$, $c_i = \cos\theta_i$ $(i = 1, 2, 3)$. Equations (101) define the matrix T, Eq. (15b). Finally, to define the matrix κ, we assume that the appendage undergoes bending vibration in the y and z directions, so that the differential operator matrix $\mathcal{L}$ for the system has the form

$$\mathcal{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathcal{L}_y & 0 \\ 0 & 0 & \mathcal{L}_z \end{bmatrix} \tag{102}$$

where, also from Ref. 1,

$$\mathcal{L}_y = \frac{\partial^2}{\partial x^2}\left(EI_y \frac{\partial^2}{\partial x^2}\right) - \frac{\partial}{\partial x}\left[\int_x^{a+L} p(\zeta,t)\,\mathrm{d}\zeta \frac{\partial}{\partial x}\right] \tag{103a}$$

$$\mathcal{L}_z = \frac{\partial^2}{\partial x^2}\left(EI_z \frac{\partial^2}{\partial x^2}\right) - \frac{\partial}{\partial x}\left[\int_x^{a+L} p(\zeta,t)\,\mathrm{d}\zeta \frac{\partial}{\partial x}\right] \tag{103b}$$

in which

$$p(x,t) = p\left[-\dot{V}_x - \omega_y V_z + \omega_z V_y + x(\omega_y^2 + \omega_z^2)\right] \tag{104}$$

In addition, the boundary conditions to be satisfied by u_y and u_z are

$$u_y = 0 \quad \text{at} \quad x = a \tag{105a}$$

$$\frac{\partial u_y}{\partial x} = 0 \quad \text{at} \quad x = a \tag{105b}$$

$$EI_y \frac{\partial^2 u_y}{\partial x^2} = 0 \quad \text{at} \quad x = a + L \tag{105c}$$

$$\frac{\partial}{\partial x}\left(EI_y \frac{\partial^2 u_y}{\partial x^2}\right) = 0 \quad \text{at} \quad x = a + L \tag{105d}$$

$$u_z = 0 \quad \text{at} \quad x = a \tag{105e}$$

$$\frac{\partial u_z}{\partial x} = 0 \quad \text{at} \quad x = a \tag{105f}$$

$$EI_z \frac{\partial^2 u_z}{\partial x^2} = 0 \quad \text{at} \quad x = a + L \tag{105g}$$

$$\frac{\partial}{\partial x}\left(EI_z \frac{\partial^2 u_z}{\partial x^2}\right) = 0 \quad \text{at} \quad x = a + L \tag{105h}$$

A. Equations for the Zero-Order Problem

We assume that the zero-order problem consist of a planar maneuver in the x,y plane, so that

$$\theta_{10} = \theta_{20} = V_{z0} = \omega_{x0} = \omega_{y0} = p_{Vz0} = p_{\omega x0} = P_{\omega y0} = 0 \tag{106}$$

Hence, Eqs. (100) reduce to

$$\tilde{S}_0 = m_1\bar{x}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{107a}$$

$$J_0 = \begin{bmatrix} J_{xx} & 0 & 0 \\ 0 & J_{yy} & 0 \\ 0 & 0 & J_{zz} \end{bmatrix} \tag{107b}$$

and Eqs. (101) to

$$C_0 = \begin{bmatrix} c_{30} & s_{30} & 0 \\ -s_{30} & c_{30} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{108a}$$

$$D_0 = \begin{bmatrix} c_{30} & s_{30} & 0 \\ -s_{30} & c_{30} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{108b}$$

Moreover,

$$\tilde{p}_{V0} = \begin{bmatrix} 0 & 0 & p_{Vy0} \\ 0 & 0 & -p_{Vx0} \\ -p_{Vy0} & p_{Vx0} & 0 \end{bmatrix} \tag{109a}$$

$$\tilde{p}_{\omega 0} = p_{\omega z0}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{109b}$$

Equations (107–109) define fully the matrices T_0^*, M_0^*, and P_0^* given by Eqs. (37), and hence the zero-order problem, Eq. (35). However, we observe that in the case at hand the problem can be compressed further. Indeed, the state vector and the associated force vector can be compressed into

$$x_0^* = \left[R_{X0} \quad R_{Y0} \quad \theta_{30} \quad p_{Vx0} \quad p_{Vy0} \quad p_{\omega z0}\right]^T \tag{110a}$$

$$X_0^{**} = \left[0 \quad 0 \quad 0 \quad F_{x0} \quad F_{y0} \quad M_{z0}\right]^T \tag{110b}$$

and the various submatrices entering into the compressed version of Eq. (36c) have the form

$$T_0^{**} = \begin{bmatrix} c_{30} & s_{30} & 0 \\ -s_{30} & c_{30} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{111a}$$

$$M_0^{**} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & m_1\bar{x} \\ 0 & m_1\bar{x} & J_{zz} \end{bmatrix} \tag{111b}$$

$$P_0^{**} = \begin{bmatrix} 0 & 0 & p_{Vy0} \\ 0 & 0 & -p_{Vx0} \\ -p_{Vy0} & p_{Vx0} & 0 \end{bmatrix} \tag{111c}$$

Moreover, it is not difficult to show that

$$(M_0^{**})^{-1} = \frac{1}{mJ_{zz} - (m_1\bar{x})^2} \times \begin{bmatrix} mJ_{zz} - (m_1\bar{x})^2/m & 0 & 0 \\ 0 & J_{zz} & -m_1\bar{x} \\ 0 & -m_1\bar{x} & m \end{bmatrix} \tag{112}$$

B. Equations for the First-Order Problem

The perturbation equations for the low-authority control are given by Eq. (54). The equation retains its form following discretization of the partial differential equations, where the perturbed state vector and associated force vector are given by Eqs. (64). It remains to derive explicit expressions for the submatrices entering into the coefficient matrix A_p. In the first place, we note that the displacement vector has the form

$$\boldsymbol{u} = \begin{bmatrix} 0 \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T \\ \boldsymbol{\phi}_y^T & \mathbf{0}^T \\ \mathbf{0}^T & \boldsymbol{\phi}_z^T \end{bmatrix}\begin{bmatrix} \xi_y \\ \xi_z \end{bmatrix} \tag{113}$$

where $\boldsymbol{\phi}_y$ and $\boldsymbol{\phi}_z$ are n vectors of space-dependent admissible functions and ξ_y and ξ_z are associated n vectors of time-dependent generalized displacements. Hence, recalling Eqs. (56), we have

$$\Phi = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T \\ \boldsymbol{\phi}_y^T & \mathbf{0}^T \\ \mathbf{0}^T & \boldsymbol{\phi}_z^T \end{bmatrix} \tag{114a}$$

$$\xi = \begin{bmatrix} \xi_y \\ \xi_z \end{bmatrix} \tag{114b}$$

We choose the components of $\boldsymbol{\phi}_y$ and $\boldsymbol{\phi}_z$ as the eigenfunctions of the fixed-free beam in bending.

One of the required submatrices is M_0, given by Eq. (60c). The matrix entries $\tilde{S}_0$ and J_0 are given by Eqs. (107). Moreover, from Eqs. (61), we can write

$$\bar{\Phi} = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T \\ \bar{\boldsymbol{\phi}}_y^T & \mathbf{0}^T \\ \mathbf{0}^T & \bar{\boldsymbol{\phi}}_z^T \end{bmatrix} \tag{115a}$$

$$\tilde{\Phi} = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0}^T & -\tilde{\boldsymbol{\phi}}_z^T \\ \tilde{\boldsymbol{\phi}}_y^T & \mathbf{0}^T \end{bmatrix} \tag{115b}$$

where

$$\bar{\boldsymbol{\phi}}_y = \int_a^{a+L} \rho \boldsymbol{\phi}_y \, \mathrm{d}x \tag{116a}$$

$$\bar{\boldsymbol{\phi}}_z = \int_a^{a+L} \rho \boldsymbol{\phi}_z \, \mathrm{d}x \tag{116b}$$

$$\tilde{\boldsymbol{\phi}}_y = \int_a^{a+L} \rho x \boldsymbol{\phi}_y \, \mathrm{d}x \tag{116c}$$

$$\tilde{\boldsymbol{\phi}}_z = \int_a^{a+L} \rho x \boldsymbol{\phi}_z \, \mathrm{d}x \tag{116d}$$

The matrix U_0 is given by Eq. (62), in which

$$\tilde{\omega}_0 \bar{\Phi} = \omega_{z0} \begin{bmatrix} -\bar{\boldsymbol{\phi}}_y^T & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix} \tag{117a}$$

$$H_0 = \int_a^{a+L} \rho \phi^T \tilde{\omega}_0 \Phi \, \mathrm{d}x = 0 \tag{117b}$$

$$H_1 = \int_a^{a+L} \rho \left(\tilde{V}_0 - \tilde{r}\tilde{\omega}_0 + \widetilde{\tilde{\omega}_0 \boldsymbol{r}} \right) \Phi \, \mathrm{d}x$$

$$= \begin{bmatrix} \mathbf{0}^T & V_{y0}\bar{\boldsymbol{\phi}}_z^T + \omega_{z0}\tilde{\boldsymbol{\phi}}_z^T \\ \mathbf{0}^T & -V_{x0}\bar{\boldsymbol{\phi}}_z^T \\ V_{x0}\bar{\boldsymbol{\phi}}_y^T & \mathbf{0}^T \end{bmatrix} \tag{117c}$$

The remaining required matrices are given by Eqs. (67). In evaluating K, Eq. (67a), we recall that the admissible functions satisfy Eq. (57), from which it follows that[4]

$$K = \int_a^{a+L} \Phi \mathcal{L} \Phi^T \, \mathrm{d}x = \Lambda \tag{118}$$

where Λ is a diagonal matrix of cantilever beam frequencies squared, $\Lambda = \mathrm{diag}[\omega_1^2 \ \omega_2^2 \ \dots \ \omega_n^2]$. Finally, Eqs. (67b–d) yield

$$H_2 = \int_a^{a+L} \rho \Phi^T \tilde{\omega}_0^T \tilde{\omega}_0 \Phi \, \mathrm{d}x = \omega_{z0}^2 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tag{119a}$$

$$H_3 = \int_a^{a+L} \rho \Phi^T \left(\tilde{V}_0 + \widetilde{\tilde{r}^T \omega_0} \right) \mathrm{d}x$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -V_{x0}\bar{\boldsymbol{\phi}}_y \\ -V_{y0}\bar{\boldsymbol{\phi}}_z - \omega_{z0}\tilde{\boldsymbol{\phi}}_z & V_{x0}\bar{\boldsymbol{\phi}}_z & \mathbf{0} \end{bmatrix} \tag{119b}$$

$$H_4 = \int_a^{a+L} \rho \Phi^T \tilde{\omega}_0^T \tilde{r} \, \mathrm{d}x = 0 \tag{119c}$$

Matrices $\mathfrak{F}_0$ and $\mathcal{E}_0$ also involve M_0^{-1} and $[M_0^{-1}]_{\omega 1}$. Because M_0 is a $(6+2n) \times (6+2n)$ matrix, the inversion process had better be left to a computer.

It should be pointed out here that, although the maneuvering problem is only of order six, the perturbation problem is of order $12+4n$.

VII. Summary and Conclusions

In an earlier paper, this author has presented the derivation of hybrid state equations of motion for flexible bodies in terms of quasicoordinates. The equations have been applied subsequently to the problem of maneuver and control of space structures. In this paper, based on the hybrid state equations in terms of quasicoordinates, a derivation of hybrid state equations in terms of quasimomenta is presented. The latter set of equations has the advantage that they are simpler in form. Then, using a perturbation approach, the hybrid state equations in terms of momenta are used to divide the formulation into one for rigid-body maneuvering and one for elastic vibration control. The perturbation approach is based on the premise that the motions defining the maneuvering are of one order of magnitude larger than the elastic motions. Consistent with this, the maneuver is carried out by a high-authority control and the elastic motions suppression is carried out by a low-authority control. The hybrid perturbation equations are then transformed into sets of ordinary differential equations through series discretization. Based on the discretized equations, an optimal control accommodating both transient and persistent disturbances is designed. As an example, the state equations for both the high-authority and low-authority control of a flexible body in a planar maneuver are derived.

Acknowledgment

This work was supported by the AFOSR Research Grant F49620-89-C-0049DEF monitored by Spencer T. Wu. The support is greatly appreciated.

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